Conservation Laws of Dynamics based on Complexity Invariance

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1. Summary

An invariant complexity, contributed by asymmetry within the pattern of forces driving motion, was identified as the source of conservation between symmetric, arbitrarily displaced trajectories at standard, relativistic, and quantum scales. A principle of causality attributing an identical outcome (constant of motion) to occurrence of a common cause (invariant force-frame) thus provides a transparent source for the conservation laws of dynamics in continuous and discontinuous processes. Causal-dynamics additionally extends the concept of randomness to asymmetry invariance in reversible order-order transitions and it furnishes an insight into the causal basis of the least action principle.

2. Introduction

A continuous symmetry of the action became indicative of a constant of motion, after Noether [1] linked energy conservation to invariance of the Lagrangian on translation in time and linear or angular momentum conservation to spatial translation or rotation. Each symmetry paired a variable with its conserved conjugate in the action. Generalizations of Noether's theorems incorporating the variational bicomplex [2, 3] or momentum map, in their Hamiltonian/symplectic form [4, 5], have enlarged the set of pairs identified and range of dynamics covered. The conservation laws of dynamics are ultimately conditional though on a quantitative correspondence between the set of independent variables that determine the pre- and post-displacement value of each constant of motion. This derives from the conceptually primitive requirement that identical causes (responsible for symmetry between pre- and post-displacement systems) produce an identical effect (constant of motion) in a deterministic physical process. Curie's symmetry principle [6] further implies that basing the conservation of a dynamical property on occurrence of a 'common cause' requires an invariance of causal asymmetry. Establishing this invariance entails deconstructing the forces driving motion into orthogonal components (causal elements) and showing their complexity is invariant in a one-to-one mapping. This provides an alternative approach to interpreting the constants of motion than that involving analysis of the Lagrangian or Hamiltonian.

Since identical causes produce the same effect, whether they are contiguous or not, the 'common cause' concept in principle applies to conservation in both continuous and discontinuous processes. The latter are beyond the scope of Noether's first theorem. Its restriction to continuous symmetries derives from a variational principle, requiring that differentiable equations define the Lie group symmetry transformations, which leave the Lagrangian invariant. Extending the calculus of variations to difference equations [7 - 9] has helped minimize this restriction. Besides avoiding the technicalities of discretizing a continuum minimization procedure, the 'common-cause'

interpretation bases the conservation laws of motion on a fundamental tenet of causality, involving a new physical invariant.

We previously examined symmetry in the pattern of forces driving motion on a geodesic path [10]. The invariant projection of the covariant derivative of the tangent vector field for a geodesic path in curved space, effectively a sequence of identical force-frames (causal elements), established that motion in the absence of an applied force follows a zero-complexity path. This reformulated Newton's first law of motion in flat space and, more generally, the least action principle as propositions of complexity theory. In this endeavor, the symmetry of the vector-frame at each point [11] on translation, rotation, reflection, and rescaling of the path of motion at standard, relativistic, and quantum scales has linked the conservation laws of dynamics to complexity invariance. The role of causal asymmetry in the order-order transitions examined here also reveals the significance of randomness in dynamical systems, beyond its long established role in order-disorder transitions.

3. Path complexity

First-, second-, and third-order time-derivatives of the equation for a path of motion, α, at arbitrary speed,

$$\begin{aligned}
\dot{\alpha} &= v\boldsymbol{e_1} \\
\ddot{\alpha} &= \dot{v}\boldsymbol{e_1} + \kappa v^2\boldsymbol{e_2} \\
\dddot{\alpha} &= (\ddot{v} - \kappa^2 v^3)\boldsymbol{e_1} + \kappa v^2\dot{v}\boldsymbol{e_2} + \kappa v^3\tau\boldsymbol{e_3}
\end{aligned} \tag{3.1}$$

yield velocity, v, curvature, κ, and torsion, τ, coefficients of the basis vectors, $\boldsymbol{e_1}$, $\boldsymbol{e_2}$, and $\boldsymbol{e_3}$, of the Frenet frame in a Euclidean space of three dimensions;

$$v = |\dot{\alpha}|$$

$$\kappa = |\dot{\alpha} \times \ddot{\alpha}| / |\dot{\alpha}|^3$$

$$\tau = (\dot{\alpha} \times \ddot{\alpha})\cdot\dddot{\alpha} / |\dot{\alpha} \times \ddot{\alpha}|^2 \qquad (3.2)$$

$$\boldsymbol{e_1} = \dot{\alpha} / |\dot{\alpha}|$$

$$\boldsymbol{e_2} = \boldsymbol{e_3} \times \boldsymbol{e_1}$$

$$\boldsymbol{e_3} = \dot{\alpha} \times \ddot{\alpha} / |\dot{\alpha} \times \ddot{\alpha}|$$

Tangent, ***T***, normal, ***N*,** and binormal, ***B*,** vectors are then

$$\boldsymbol{T} = v\boldsymbol{e_1}$$
$$\boldsymbol{N} = \boldsymbol{B} \times \boldsymbol{T} \qquad (3.3)$$
$$\boldsymbol{B} = (v\boldsymbol{e_1}) \times (\dot{v}\boldsymbol{e_1} + \kappa v^2\boldsymbol{e_2})$$

T, ***N*, *B*** share the same point of application in the path, making them orthogonal, with a resultant vector, $\boldsymbol{V_R} = \boldsymbol{T} + \boldsymbol{N} + \boldsymbol{B}$, of magnitude, $|\boldsymbol{V_R}| = ((\boldsymbol{T}\cdot\boldsymbol{T}) + (\boldsymbol{N}\cdot\boldsymbol{N}) + (\boldsymbol{B}\cdot\boldsymbol{B}))^{½}$. Their rate of change is

$$\dot{\boldsymbol{T}} = \kappa v\boldsymbol{e_2}$$
$$\dot{\boldsymbol{N}} = -\kappa v\boldsymbol{e_1} + \tau v\boldsymbol{e_3} \qquad (3.4)$$
$$\dot{\boldsymbol{B}} = -\tau v\boldsymbol{e_2}$$

in terms of the basis vectors. When combined with a constant scalar mass term, $\dot{\boldsymbol{T}}$, $\dot{\boldsymbol{N}}$, $\dot{\boldsymbol{B}}$ are components of the force producing motion in standard dynamics. Decomposing $\boldsymbol{V_R}$ into ***T***, ***N*,** and ***B*** opens the way to showing the constants of motion result from occurrence of a common pattern of forces (common cause). This proposition implies that the conservation laws of dynamics derive from symmetry between relevant dynamical states, illustrated by displacement of a path of motion. Agreement between time-derivatives of the path equation defining pre- and post-displacement *TNB*-frames will affirm this symmetry. Occurrence of the symmetry implies invariance of path complexity contributed by asymmetry in the pattern *TNB*-frame vectors and, by extension, in the pattern of forces driving motion in each state.

A set of vector magnitudes, $\frac{(\boldsymbol{T.T})^{1/2}}{|\boldsymbol{V}|}$, $\frac{(\boldsymbol{N.N})^{1/2}}{|\boldsymbol{V}|}$, $\frac{(\boldsymbol{B.B})^{1/2}}{|\boldsymbol{V}|}$, normalized with respect to the sum of their numerators, $|\boldsymbol{V}|$**,** indicates the comparative contribution of each 'velocity' to the *TNB* (momentum) frame. Asymmetry within the frame has complexity density

$$C_{\text{int}}(\text{p}) = -\sum_{\text{i}=1}^{3} \varphi_{\text{i}} \ln \varphi_{\text{i}} \tag{3.5}$$

per component. It provides a single-valued, positive measure of intrinsic complexity, based on the ratios, φ_{i} (i = 1, 2, 3). Within-frame complexity vanishes, as required, when there is a single vector ($\varphi_{\text{i}} = 1$), and it is maximized when $\varphi_1 = \varphi_2 = \varphi_3$ [10]. φ_{i} is redefined to accommodate other frames. Ratios formed with respect to the squared magnitude of the resultant vector, $\frac{(\boldsymbol{T.T})}{|\boldsymbol{V_R}|^2}$, $\frac{(\boldsymbol{N.N})}{|\boldsymbol{V_R}|^2}$, $\frac{(\boldsymbol{B.B})}{|\boldsymbol{V_R}|^2}$, yield weighting terms indicative of the energy distribution within a specified frame in the path of motion. Their rate of change deduced from the $|\text{V}_\text{R}|^2$ expression yields terms, $\frac{\boldsymbol{T\dot{T}}}{|\boldsymbol{V_R}||\boldsymbol{\dot{V}_R}|}$, $\frac{\boldsymbol{N\dot{N}}}{|\boldsymbol{V_R}||\boldsymbol{\dot{V}_R}|}$, $\frac{\boldsymbol{B\dot{B}}}{|\boldsymbol{V_R}||\boldsymbol{\dot{V}_R}|}$, related to complexity density in the force-frame. It is apparent that path equation time-derivatives to third-order are sufficient to quantitate asymmetry within the pattern of forces driving motion. Identical time-derivatives in arbitrarily displaced trajectories thus establishes complexity invariance in their respective force-frames, and this links the constants of motion resulting from path symmetry to occurrence of a 'common cause'.

Use of a generalized entropy to measure dynamical complexity [10, 12-14] implies causal asymmetry is a form of randomness in a physical system. In this context, Curie's symmetry principle [6] requires that path-complexity cannot exceed complexity within the *TNB*-frame. Conversely, intrinsic complexity, represented by vector asymmetry, cannot exceed the complexity of the path of motion from which it was determined.

Theorem 3.1. *The equation for the path of motion can be recovered from the TNB frame at each point in the path of motion.*

Proof. From definition of the tangent vector, $\boldsymbol{T} = v\boldsymbol{e}_1$, integration over a path interval, t,

$$\int_t \mathrm{dt}\ v\boldsymbol{e_1} = a, \quad \text{where, } v\boldsymbol{e_1} = \dot{\boldsymbol{a}} \tag{3.6}$$

restores the path equation, *a*.

Lemma 3.2. *A path and its TNB frame have equal complexity at any point.*

Proof. Time derivatives of the path equation specify the *TNB*-frame at any point (equation 3.3). Reciprocally, the path equation can be recovered on integration of the tangent vector in the *TNB*-frame (Theorem 3.1). Since each transformation reverses the other, they involve a one-to-one (bijective) mapping. The complexity based on causal asymmetry is, consequently, conserved in these transformations [6, 10]. ■

Invariance of the *TNB*-frame at equivalent points in inter-path comparisons, following an arbitrary displacement of a dynamical system, with its path of motion and driving forces, implies invariance of intrinsic complexity at every point over the whole path. Asymmetry in the distribution of *TNB*-frames, produced by changes in vector weight over the path, yields a global complexity

$$C_{\mathrm{ext}} = -\int_{\mathrm{j(a)}} \mathrm{dt}\ \Big(\sum_{\mathrm{i=1}} \varphi_{\mathrm{i}}(\mathrm{t}) \ln \varphi_{\mathrm{i}}(\mathrm{t}) \Big) \tag{3.7}$$

per *TNB*-frame component, in the $\mathrm{j^{th}}$ path segment, and is termed the extrinsic path complexity. $\varphi_{\mathrm{i}}(\mathrm{t})$ is vector-i weight at time, t (equation 3.5). Integration is over the $\mathrm{j^{th}}$ path segment, from end-points j(a) to j(b). For a path of length, L, whose $\mathrm{j^{th}}$ segment has length, ℓ_{j}, $L = \Sigma_{\mathrm{j}}\, \ell_{\mathrm{j}}$ (j = 1, 2, …, n), where

$$\ell_{\mathrm{j}} = \int_{\mathrm{j(a)}}^{\mathrm{j(b)}} \mathrm{dt}\ |\dot{\alpha}| \tag{3.8}$$

$|\dot{\alpha}|$ being the magnitude of the path equation time-derivative. A path represented by a continuous function of degree *n*, has n zeros and a maximum of *n* – 1 turning points. The latter divide the path into *n* arc-segments. Symmetry between segments decreases path complexity. The zeros of the minimal polynomial *p*(*x*) define a related measure, the topological complexity

$$m(p) = \log M(p)$$

$$M(p) = |a|\prod_{i=1}^{n} \max\{1, |\alpha_i|\} = |a| \prod_{|\alpha_i| \geq 1} |\alpha_i| \tag{3.9}$$

of a curve [15, 16]; where a denotes the leading coefficient, α_i are polynomial zeros, and $m(p)$ is the logarithmic Mahler measure.

A *TNB*-frame with an invariant constant-length tangent vector produces a straight particle world-line in flat space. A single segment path results, such that $\ell_1 = L$. With a path equation of degree-1 and zero path complexity, this state may be noted to correspond to a primary dynamical state, according to Newton's first law.

Pre-and post-displacement paths of motion are shown below to have identical time-derivatives to third order, demonstrating vector-frame invariance on displacement. In an approach analogous to that adopted by Feynman to relate symmetry to the constants of motion [17], this result serves here to link the conservation laws of dynamics to complexity invariance within forces driving motion.

4. Complexity Invariance with Path Displacement

For generality, the path equation is taken to be a function of degree n,

$$\alpha = a_n t^n + a_{n-1} t^{n-1} + a_{n-2} t^{n-2} + \ldots + a_0 \tag{4.1}$$

dependent on time, t, with coefficients a_i. Most physical interactions can be expected to involve trajectories with an equation of low degree, while biophysically produced motion could be complex and require an equation of high degree. The required time-derivatives are

$$\begin{aligned}
\dot{\alpha} &= n a_n t^{n-1} + (n-1) a_{n-1} t^{n-2} + (n-2) a_{n-2} t^{n-3} + \ldots + a_1 \\
\ddot{\alpha} &= n(n-1) a_n t^{n-2} + (n-1)(n-2) a_{n-1} t^{n-3} + (n-2)(n-3) a_{n-2} t^{n-4} + \ldots + a_2 \\
\dddot{\alpha} &= n(n-1)(n-2) a_n t^{n-3} + (n-1)(n-2)(n-3) a_{n-1} t^{n-4} + (n-2)(n-3)(n-4) a_{n-2} t^{n-5} + a_3
\end{aligned} \tag{4.2}$$

Translation of the path, $(\alpha, t) \Longrightarrow (\alpha + b, t - c)$, yields the equation,

$$\alpha = a_n u^n + a_{n-1}u^{n-1} + a_{n-2}u^{n-2} + \ldots + (a_0 + b),\ \ u = t_0 = (t - c) \quad \begin{cases} t > 0, c > 0 \\ t < 0, c < 0 \end{cases} \tag{4.3}$$

with time-derivatives,

$$\begin{aligned} \dot{\alpha} &= na_n u^{n-1} + (n-1)a_{n-1}u^{n-2} + (n-2)a_{n-2}u^{n-3} + \ldots + a_1,\ \ u' = 1 \\ \ddot{\alpha} &= n(n-1)a_n u^{n-2} + (n-1)(n-2)a_{n-1}u^{n-3} + (n-2)(n-3)a_{n-2}u^{n-4} + \ldots + a_2 \\ \dddot{\alpha} &= n(n-1)(n-2)a_n u^{n-3} + (n-1)(n-2)(n-3)a_{n-1}u^{n-4} + \\ &\qquad (n-2)(n-3)(n-4)a_{n-2}u^{n-5} + \ldots + a_3 \end{aligned} \tag{4.4}$$

in terms of the variable, u. Its time-derivative, $\dot{u} = 1$, reveals that, $du = dt$. With time-derivatives in the translated path unchanged (equations 4.2, 4.4), ***T***, ***N***, and ***B*** vectors remain invariant (equation 3.3). Pre- and post-translated *TNB*-frames consequently relate one-to-one. Complexity density and its global level (equations 3.5, 3.7) are consequently conserved on path translation.

Path rotation by ϕ transforms equation (4.1) to yield,

$$(\alpha, t) \equiv (r\sin\theta, r\cos\theta) \Longrightarrow (r\sin(\theta + \phi), r\cos(\theta + \phi))\ ,\ 0 < (\theta + \phi) < \frac{\pi}{2}$$

$$\begin{aligned} \alpha &= r(\sin\theta\cos\phi - \cos\theta\sin\phi) = \alpha_i\cos\phi - t_i\sin\phi;\ \text{where, } \alpha_i \equiv r\sin\theta \\ &\lim_{\phi\to 0} \frac{d\alpha}{dt}\frac{dt}{dt_i} = \dot{\alpha}_i\cos\phi \\ t &= r(\cos\theta\cos\phi - \sin\theta\sin\phi) = t_i\cos\phi - \alpha_i\sin\phi;\ \text{where, } t_i \equiv r\cos\theta \\ &\lim_{\phi\to 0} \frac{dt}{dt_i} = \cos\phi \end{aligned} \tag{4.5}$$

where a subscript-i designates an initial parameter. Cancellation of cos ϕ, in the third equation (based on the fifth relation) yields the pre-rotation time-derivatives (equation 4.2). Thus, intrinsic and extrinsic complexity remain invariant on path rotation.

A path formed by parallel tangent vectors produced on reflection from an inelastic surface maps one-to-one, after allowing for rotation at the surface, with pre-collision path vectors. Equation (4.5)

can then be restated as,

$$\alpha = a_1 t + a_0\,;\;\; t_{in} = r\cos\theta \Longrightarrow t_{out} = r\cos(\pi - \theta) = -\,r(t)\cos\theta,\; 0 < \theta \leq \pi$$

$$\begin{aligned} &\alpha_{out} = -\alpha_{in} = -a_1 \\ &\dot{\alpha} = 0 \\ &\ddot{\alpha} = 0 \qquad\qquad (4.6) \\ &\dddot{\alpha} = 0 \end{aligned}$$

$r(t)$ and θ denote radial distance and angle of incidence, respectively. Reflection at time, t_R, rotates the path from θ to $(\pi - \theta)$ radians. A negative slope results with increasing time. A velocity of constant magnitude yields,

$$|\dot{\alpha}| = a_1 \quad \begin{cases} t_{in} = r(t)\cos\theta, & t < t_R \\ t_{out} = r(t)\cos(\pi - \theta), & t > t_R \end{cases}$$

tangent vectors of constant magnitude, $a_1\boldsymbol{e_1}$, in the path of motion. As pre- and post-reflection path *TNB*-frames map one-to-one, inward and outward path-complexities are identical.

A rescaling factor, ζ, modifies the path (equation 4.1) to yield,

$$\beta = a_n\,\zeta\, t^n + a_{n-1}\,\zeta\, t^{n-1} + a_{n-2}\,\zeta\, t^{n-2} + \ldots + \zeta\, a_0\,,\;\; \beta = \zeta\alpha \quad \begin{cases} \text{dilation} & \zeta > 1 \\ \text{compression} & \zeta < 1 \end{cases}$$

$$\begin{aligned} &\dot{\beta} = \zeta\,\dot{\alpha} = n\,\zeta\, a_n t^{n-1} + (n-1)a_{n-1}\,\zeta\, t^{n-2} + (n-2)a_{n-2}\,\zeta\, t^{n-3} + \ldots + a_1\,\zeta, \\ &\dot{\alpha} = \mathrm{n}a_n t^{n-1} + (n-1)a_{n-1}t^{n-2} + (n-2)a_{n-2}t^{n-3} + \ldots + a_1 \qquad\qquad (4.7) \\ &\ddot{\alpha} = n(n-1)a_n t^{n-2} + (n-1)(n-2)a_{n-1}t^{n-3} + (n-2)(n-3)a_{n-2}t^{n-4} + \ldots + a_2 \\ &\dddot{\alpha} = n(n-1)(n-2)a_n t^{n-3} + (n-1)(n-2)(n-3)a_{n-1}t^{n-4} + (n-2)(n-3)(n-4)a_{n-2}t^{n-5} + \ldots + a_3 \end{aligned}$$

Pre-scaling time-derivatives (equation 4.2) are recovered, on cancelling γ in the second equation. Path-complexity is thus preserved on re-scaling.

Symmetry of the path of motion on translation, rotation, reflection, and rescaling establishes that

TNB-frames and path topology remain invariant with arbitrary displacement.

5. Conservation laws

(a) Standard dynamics

Invariance of the velocity in the path of motion following translation, rotation, reflection, dilation, or compression (§4) establishes that the linear momentum, $\boldsymbol{p}$, of a moving body is conserved,

$$\boldsymbol{p} = m|\dot{\alpha}|\boldsymbol{e}_1 \tag{5.1}$$

in a Gallilean system, with constant mass, m. $|\alpha|$ and $\boldsymbol{e}_1$ refer to the magnitude of the velocity and Frenet tangent vector, respectively (equations 3.1, 3.3).

An applied force contributes tangential and curvature terms to the velocity (equation 3.1-3.4),

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\alpha}\boldsymbol{e}_1) = \ddot{\alpha}\boldsymbol{e}_1 + \dot{\alpha}\dot{\boldsymbol{e}}_1 = \dot{v}\boldsymbol{e}_1 + \kappa v^2\boldsymbol{e}_2 \tag{5.2}$$

from the time-derivative of the momentum; suppressing mass. Results in §4 reveal that v, $\dot{v}$, and κ map one-to-one at each point within pre- and post-displaced paths. $\boldsymbol{p}$ values in response to an applied force, therefore, agree at corresponding points within both paths. Invariant *TNB*-frames ensure conservation of linear momentum on path displacement.

Angular momentum, $\boldsymbol{L}$,

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}\ ,\ |r| = \frac{1}{\kappa} \tag{5.3}$$

acts normal to the Frenet tangent vector, $\boldsymbol{e}_1$, curving the path of motion. $\boldsymbol{r}$ and $|r|$ denote the radial vector and its magnitude; the latter depends inversely on the curvature factor (equation 3.1), $\kappa = |r|^{-1}$. An invariant second order time-derivative of the path equation constrains κ to its initial value

during path displacement (§4). Angular momentum, like linear momentum, is therefore conserved under path displacement. Invariance of the time-derivatives (equation 4.2) in these transformations also fixes the Darboux vector, $\boldsymbol{\omega}$,

$$\boldsymbol{\omega} = \tau\boldsymbol{e}_1 + \kappa\boldsymbol{e}_3 \tag{5.4}$$

combining orbital, κ, and spin, τ, momentum terms, in a manner proportional to the total path angular momentum.

A tangentially directed force exerted over a distance, $\boldsymbol{r}_1$, $\boldsymbol{r}_2$,

$$\begin{gathered}\int_1^2 \boldsymbol{p}.\mathrm{d}\boldsymbol{v} = \int_1^2 \boldsymbol{F}.\mathrm{d}\boldsymbol{r},\ \oint \boldsymbol{F}.\mathrm{d}\boldsymbol{r} = 0\,,\ \int_r \boldsymbol{F}.\mathrm{d}\boldsymbol{r} = -\nabla U(\mathbf{r}) \\ \tfrac{1}{2}\,(m_2\boldsymbol{v}_2^{\,2} - m_1\boldsymbol{v}_1^{\,2}) = U(\boldsymbol{r}_1) - U(\boldsymbol{r}_2) \implies K_1 + U(\boldsymbol{r}_1) = K_2 + U(\boldsymbol{r}_2)\end{gathered} \tag{5.5}$$

performs work equal to the increase in kinetic energy ($K_1 \rightarrow K_2$), evident on integrating momentum with respect to changes in velocity, d$\boldsymbol{v}$. Tha amount of work is path-independent (second expression) and equal to the gradient of the potential energy, $U(\boldsymbol{r})$. Symmetry between initial (I) and displaced (D) paths of motion (§4) requires both have an equal amount of energy at equivalent path locations. Vanishing of the force difference between both paths on integration over a specified distance, $\boldsymbol{r}$, thus provides an energy conservation condition,

$$\begin{gathered}\int_r (\boldsymbol{F}(\mathrm{D}) - \boldsymbol{F}(\mathrm{I})).\mathrm{d}\boldsymbol{r} = 0 \\ \sum_{\mathrm{i}} \int_r (\boldsymbol{F}_{\mathrm{i}}(\mathrm{D}) - \boldsymbol{F}_{\mathrm{i}}(\mathrm{I})).\mathrm{d}\boldsymbol{r} = 0\end{gathered} \tag{5.6}$$

Contributions of each *TNB*-frame component are summed in the second expression. Global path symmetry (§4) allows $\boldsymbol{r}$ to extend over the whole path.

A comparable condition holds for momentum conservation,

$$\int_t (\boldsymbol{F}(\mathrm{D}) - \boldsymbol{F}(\mathrm{I}))\mathrm{d}t = 0 \tag{5.7}$$

$$\sum_i \int_t (\boldsymbol{F}_i(\mathrm{D}) - \boldsymbol{F}_i(\mathrm{I}))\mathrm{d}t = 0$$

where, $\boldsymbol{F} = \dot{\boldsymbol{p}}$, implying that $\int_t \boldsymbol{F}\mathrm{d}t = m \int_t \mathrm{d}\boldsymbol{v}$ for linear momentum. For rotational momentum, based on Darboux's vector, $\int_t \boldsymbol{F}\mathrm{d}t = m \int_t \mathrm{d}\boldsymbol{\omega}$. A standard dynamical system that conforms with equations (5.6, 5.7) thus has constants of motion, based on inter-path symmetry and complexity invariance (§4).

(b) Relativistic dynamics

An inertial frame, F', with a relativistic boost, υ_1, on the first spatial coordinate and zero on the remaining two spatial coordinates ($\upsilon_2 = \upsilon_3 = 0$), has transforms

$$\begin{aligned} \Delta_0' &= \gamma(\Delta_0 - \upsilon_1\Delta_1/c^2)\ , \quad \gamma = \left(1 - \frac{\upsilon_1^2}{\mathrm{c}^2}\right)^{-1/2} \\ \Delta_1' &= \gamma(\Delta_1 - \upsilon_1\Delta_0) \\ \Delta_2' &= \Delta_2 \\ \Delta_3' &= \Delta_3 \end{aligned} \tag{5.8}$$

for the interval, Δ_0', and distances, Δ_i' (i = 1, 2, 3), in the path of motion, based on lab frame, F, measurements; γ being the Lorentz transform. Addition of the resulting velocities

$$\dot{a}' = \frac{\dot{a} - \upsilon_1}{(1 - \dot{a}\upsilon_1/\mathrm{c}^2)} \tag{5.9}$$

is frame-dependent: a speed of $\dot{a}'$ in F' being $(\dot{a} - \upsilon_1)$ in F, and slower by a factor of $(1 - \dot{a}\upsilon_1/\mathrm{c}^2) < 1$.

A restriction of bodies with mass to sub-luminal speeds, $a <$ c *in vacuo*, results in the non-linear addition of velocities [18]. Path symmetry on displacement, apparent in a Galilean system (§5a), accordingly, does not hold when boost speeds approach c, in the quasi-Euclidean, four-dimensional

Minkowski space of special relativity. Merging energy with momentum in the relativistic four-momentum also modifies the constants of motion.

The four-momentum

$$\boldsymbol{p} = m\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}T} = [\frac{E}{\mathrm{c}}, p_1, p_2, p_3], \ \ \Delta\boldsymbol{r} = (\mathrm{c}^2\Delta_0^{\,2} - \Delta_1^{\,2} - \Delta_2^{\,2} - \Delta_3^{\,2})^{½}, \quad \mathrm{d}T = \frac{\mathrm{d}t}{\gamma} \tag{5.10}$$

obtained with the proper four-vector, $\boldsymbol{r}$, and proper time, T, restores frame invariance and path symmetry. dt refers to an infinitesimal time elapsed in the lab frame. γ retains its former meaning. Intervals, Δ_i, are from equation (5.8). p_i (i = 1, 2, 3) are momenta in each spatial direction, $\boldsymbol{e}_\mathbf{i}$, and m is the proper (rest) mass. E refers to total energy. Adopting frame-invariant parameters of motion restores conservation of the four-momentum at equivalent points in symmetric paths (§4).

A frame-invariant expression for the energy follows,

$$\boldsymbol{p}\cdot\boldsymbol{p} = \frac{E^2}{c^2} - (p_1^{\,2} + p_2^{\,2} + p_3^{\,2})$$
$$E^2 - p^2\mathrm{c}^2 = m^2\mathrm{c}^4 \Longrightarrow E = ((p\mathrm{c})^2 + (m\mathrm{c}^2)^2)^{½} \tag{5.11}$$

on equating the scalar product of the four-momentum with the static (non-kinetic) component of the energy squared. Conservation of the four-momentum and fixed rest-mass ensures that total energy (positive root) is conserved at equivalent points in symmetric paths of motion. This applies when both paths satisfy the difference conditions in equations (5.6, 5.7). Conservation of momentum and energy during motion on symmetric paths thus hold, following a relativistic scale boost.

Energy-momentum conservation in the relativistic theory of gravity generally requires a flat (Minkowski) or asymptotically flat (Schwarzchild) space-time. This departure from the conservation laws of motion involves the stress-energy tensor, $\boldsymbol{T}_{\mu\upsilon}$, of the Einstein-Hilbert field equation in a Riemann space,

$$\boldsymbol{R}_{\mu\upsilon} - \tfrac{1}{2}\, \boldsymbol{g}_{\mu\upsilon} R = \frac{8\pi G}{c^4}\, \boldsymbol{T}_{\mu\upsilon} \qquad (5.12)$$

$\boldsymbol{R}_{\mu\upsilon}$ being Ricci's curvature tensor, R curvature scalar, $\boldsymbol{g}_{\mu\upsilon}$ metric tensor, and G gravitational constant, with the cosomological constant suppressed ($\Lambda = 0$). Local $\boldsymbol{T}_{\mu\upsilon}$, defined in asymptotically similar tangent spaces, show tensor character. Local energy-momentum conservation [1] is thus demonstrable. As tensors defined in different tangent spaces cannot be added or subtracted, $\boldsymbol{T}_{\mu\upsilon}$ lack tensor character in a globally curved space-time. Hence, the difference conditions (equations 5.6, 5.7), establishing force-frame invariance on path displacement, is not met. For arbitrarily displaced paths of motion spanning different tangent spaces, therefore, energy-momentum conservation cannot be demonstrated. An isometric displacement of the path of motion, embedded in a curved space-time may be anticipated to yield identical $\boldsymbol{T}_{\mu\upsilon}$ at equivalent points in both paths and conserve inter-path energy-momentum and complexity.

(c) Quantum dynamics

Motion on the path of least action [19] corresponds to the most probable path, among the set of all possible paths, in the path-integral formulation of quantum dynamics [20];

$$\begin{aligned} &U(q(0); q(t)) = \langle q(t)|\hat{U}(t)|q(0)\rangle = \mathcal{A} \sum_{\text{all paths}} \exp[-iS/\hbar] \\ &S = \int_t \mathcal{D}(t)\, \mathcal{L}(q(t), \dot{q}(t)) \end{aligned} \qquad (5.16)$$

$\hat{U}(t)$ is the time evolution operator giving the Feynman amplitude, U(q(0);q(t)), for the transition between quantum states q(0) and q(t), where 0 and t are path end-points. S denotes the action for particle motion on a given path. The Lagrangian, $\mathcal{L}$, is integrated over all slices of a path. $\mathcal{D}(t)$ indicates that integration, like the summation (path integral), applies to multiple slices of individual paths in the set of all paths. $\mathcal{A}$ is a normalization constant. Phase factors in paths, among all possible

paths, whose action exceeds the minimum by more than π/ħ (ħ, reduced Planck constant) usually interfere destructively, through a superposition of states, and vanish.

Results linking constants of motion during standard dynamics to invariant force-frames under path displacement (§5.1) apply directly to the subset of quantum paths of motion revealed by the path integral formulation of quantum mechanics to conform with the least action principle. To the extent that quantum motion departs from the canonical path, it incorporates dynamical 'noise' from random fluctuations. They introduce a quantum-scale path entropy that reduces the maximum attainable path complexity [10].

6. Discussion

Attributing the constants of motion to an invariant pattern of the forces driving motion made it possible to place conservation in continuous and discontinuous processes on an equal theoretical footing. Difference equations [7] were accordingly not required for the latter. As arbitrarily displaced paths retain initial vector-frames (§4), the resulting symmetry established that path-complexity, contributed by force-frame asymmetry, remained invariant. This validated the proposition that occurrence of a 'common cause' underlies the constants of motion at standard and relativistic scales, and at quantum scale on the most probable path.

The concept of a 'common cause' expands the role of symmetry in conservation beyond that of a single element of symmetry. In contrast to crediting the constants of motion to symmetry of the conjugate variable in the action, on an infinitesimal displacement of a Lagrangian or Hamiltonian system [1, 17], the 'common cause' principle presupposes that all independent variables determining a constant of motion are invariant. By basing the constants of motion on occurrence of a 'common cause', in place of the 'variable conjugate to the symmetric variable of the action' notion, complexity theory provides a transparent explanation for the conservation laws of dynamics.

Compliance with Euler's equation [19] establishes that Lagrangian dynamics are subject to a least action principle. In this event,

$$\frac{d}{dt}\frac{\partial \boldsymbol{L}}{\partial \dot{\boldsymbol{r}}} = \frac{\partial \boldsymbol{L}}{\partial \boldsymbol{r}}, \text{ where } \boldsymbol{L} = K(\dot{\boldsymbol{r}}) - U(\boldsymbol{r})$$

applying notation from equations (5.5, 5.16). The left-hand term corresponds to a Newtonian force, $\frac{dK(\dot{\boldsymbol{r}})}{dt}$, while the right-hand term will be recognized (equation 5.5) as the decrease in potential energy. A first-order rate of change in velocity, $d\dot{\boldsymbol{r}}/dt$, in response to a force, implies the transition between dynamical states, with velocities $\dot{\boldsymbol{r}}_1$ and $\dot{\boldsymbol{r}}_2$, proceeds by a straight path in phase space ($\dot{\boldsymbol{r}}$, t). Symmetry produced by the constant rate of increase, $\ddot{\boldsymbol{r}}_1$, in the transition between neighboring frames, minimizes transition path complexity, consistent with it being the path of shortest length (geodesic) [10]. In a conservative dynamical system, reversing the direction of the transition operation restores the initial state. This links the least action principle to complexity invariance and occurrence of a common cause.

As the constants of motion were shown to imply the invariance of causal asymmetry (constant force-frame complexity), the theory of causal-dynamics entails a quantitative extension of Curie's principle [6] to reversible order-order transitions (§4). Use of a generalized entropy measure in this context (equations 3.5, 3.7) portrays complexity as a second form of physical randomness within a dynamical system [12-14, 21, 22]. The exclusion of deterministic increases in causal asymmetry by Curie's principle in reversible order-order transitions [6] can be consequently viewed as a dynamical analogue of Boltzmann's principle, excluding spontaneous decreases in statistical entropy within an isolated thermodynamic system.

Information theory already distinguishes between statistical entropy and information. Entropy being the microstate information within a statistical mechanical system that is inaccessible to a macroscopic observer [23], or randomness present. While information refers to the amount of

statistical entropy removed on receipt of a communication [24], or randomness removed. As a measure of causal asymmetry, by contrast, complexity quantitates the degree of 'mixing' among different causal elements in a deterministic process. Complexity based on aperiodicity corresponds, with a suitable choice of units, to complexity based on the length of a minimal algorithm [21, 22].

Applying complexity theory to motion has led to findings that lay the basis for a theory of causal-dynamics. The consistency of this theory with conventional principles may be illustrated with a primary dynamical state, involving force-free motion (equations 3.5, 3.7, 5.16). Parallel tangent vectors of constant length form the vector-frame at each point of a geodesic, in a flat or curved space [10]. A trajectory produced by a single-component vector-frame has zero complexity density. Since a single homogeneous segment then spans the full length of the path, such that $\ell_1 = L$ (§3), global path complexity is also zero (equation 3.7). The world line of a body at rest is equally time-invariant. Both primary dynamical states enumerated by Newton's first law thus represent states with zero path-complexity paths.

A circular path has an infinite number of symmetries and single generator ($\ell_1 = L$) of a unitary Lie group, $U(1)$, resulting in zero extrinsic complexity (equation 3.7). In view of this, circular motion may appear to qualify as a primary dynamical state, at variance with Newton's first law. Although invariant, its *TNB*-frame is bivectorial (axial, tangential components) and this contributes an initial asymmetry resulting in a non-zero intrinsic complexity (equation 3.5). As a consequence, circular motion is neither a force-free nor complexity-free state. This outcome furthers the correspondence between causal-dynamics and the laws of motion.

7. Conclusion

The conservation laws of dynamics result from occurrence of a common cause at standard and relativistic scales, and on the most probable path of motion at quantum scale. All variables

determining each constant of motion remain invariant under translation, rotation, reflection, and rescaling of the trajectory. The role of symmetry in conservation therefore extends beyond the single element of symmetry linked to each constant of motion by analysis of the Lagrangian or Hamiltonian. Conservation is empirically and conceptually equivalent in continuous and discontinuous processes, as each require occurrence of a common cause. Complexity contributed by asymmetry in the pattern of forces driving motion is an invariant in the reversible order-order transitions of dynamics. Causal-dynamics thus expands the role of randomness in physical processes, beyond the long established action of statistical entropy in order-disorder transitions.

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